‑

Euclidean Rhythms: The Relationship between Mathematics & Music

Kai (Jasmine) White

Senior Investigative Paper

In fulfillment of Johnson C. Smith University



Dr. Christopher Weise, Second Reader

Abstract

Euclidean rhythms are evidence of the many links between music and mathematics. Euclidean rhythms are developed from Euclid’s algorithm and Bjorklund’s algorithm. Since the first journal article written by Godfried Toussaint in 2005, musicians and engineers have found many uses of Euclidean rhythms and the different ways to create them. These methods have been expressed using circles which show the even distance between the rhythm and binary code which shows the division of the beats and rest. More recently, engineers have coded the Euclidean rhythm into patches that are used in music production. A patch is a group of settings on a musical synthesizer or sampler that alters the input of a note resulting in the desired output. For example, a picture is taken (input), a filter is added (patch) and the newly filtered picture is the output. There are multiple types of Euclidean rhythms and they can be used in many musical genres.

Introduction

Music and mathematics have many connecting topics that span across centuries although most people believe they are not typically subjects that most think go together. Euclidean rhythms is one of the more modern connecting topics that have been discovered. I am interested in this topic because it is a unique way to create rhythms. The first evidence of this discovery was in 2005 so it’s still a relatively new topic. It also links original mathematical ideas to new musical concepts showing the relevancy mathematics and music will always have. Euclidean rhythms are generated using the Euclidean algorithm which computes the greatest common divisor of two integers (Toussaint 2005). Euclidean rhythms are important in a few different ways and can be used in multiple rhythms from around the world which means that Euclidean rhythms are versatile. The difference between the rhythms are based on the complexities between the pulses and the steps. Pulses, in musical terms, are the total amount of beats in a rhythm. For the sake of this topic, they are better described as the total number of places an event can happen. For example, if there are 13 pulses in a Euclidean rhythm, there are 13 possible places that a step (also known as onsets) can be placed. A step (or event) is the actual audible part of the rhythm and the rests (or non-event) are the parts of the rhythm that are silent. For example, if there are 13 pulses and 9 steps, there would be 4 beats of silence. There will be an analysis of the evidence of math in music shown through the Euclidean rhythm.

The purpose of this analysis is to examine the relatively recent link between mathematics and music. The research questions involve further investigating Euclidean rhythms. The first research question is: what is the history of Euclidean rhythms and how they have evolved over time? The history is important because it allows the researcher to evaluate Euclidean rhythms in their purest form and how they have changed through the years. The next question is: what is the relationship between mathematics and music with Euclidean rhythms? This question is the basis of the research. The last question is: what genres or styles cannot be obtained using Euclidean rhythms? The purpose of this question is to allow for a well-rounded analysis of the different types of Euclidean rhythms.

We will begin by reviewing the literature about Euclidean rhythms and then discuss the history of Euclidean rhythms and introduce another concept derived from the Euclidean Algorithm, Euclidean strings. We will also explore the similarities between Euclidean strings and Euclidean rhythms. Finally, we will evaluate the different operations that can be performed on Euclidean rhythms to alter their properties.

Literature Review

“The Distance Geometry of Music” discusses the Euclidean Algorithm and how it relates to different fields of study. The authors use the Euclidean Algorithm to generate over forty different rhythms used around the world. The rhythms all have equal distances as related to a circle and this is a way to show the mathematical properties of Euclidean rhythms and these different beats from around the world. This source was very helpful in showing the mathematics of Euclidean rhythms.

“The Euclidean Algorithm Generates Traditional Musical Rhythm” discusses the origin of the Euclidean algorithm and how it is derived from Euclid’s Elements is explained. Euclid’s Elements is a collection of 13 books written by the mathematician Euclid in 300 BC. In these books, Euclid provides definitions and proofs from different math subjects such as geometry and algebra. Toussaint specifies the evenness of African music but also delves into other world music rhythms. Using Euclidean algorithms can result in another phenomenon called Euclidean strings which are compared to Euclidean rhythms in this article. The comparison between the two will be discussed further in the paper.

In the article "Kid Algebra: Radiohead's Euclidean and Maximally Even Rhythms”, Olson describes the four types of Euclidean rhythms. They are mathematically written as E(*k*,*n*) where *k* and *n* are nonnegative integers, *k* is the onset and *n* is the number of elements being divided by the onset. The first type is when *k* is a factor of *n* so it divides *n* evenly. The second type is when *k* is not a factor of *n* and they are also not coprime (when the only positive integer that divides both the numbers is one). The third type is when *k* is not a factor of *n* where *k* and *n* are coprime; *k* is not a factor of [n±1]. The fourth type is when *k* is not a factor of *n* where *k* and *n* are coprime; and *k* is a factor of [n±1]. Olson then uses this information to breakdown different Radiohead songs to discover that they all use one of these types. The information about Radiohead will not be used because it is not relevant to the research but the different types of Euclidean rhythms provide useful information for this analysis.

“Interlocking and Euclidean Rhythms” explains the different variations of Euclidean rhythms through “performing operations” on them. Operations are mainly used for composing, analyzing music, and improvising solos. The three operations discussed are complementation, alternation, and decomposition. Finally, the connection is made to music where the operations are explained using onsets and pulses.

“Euclidean Rhythms: An Investigation into the Structure of Maximal and General Evenness of Rhythms” explains two different classes of Euclidean rhythms. The first class occurs when the rhythm is maximally even while the second class is derived by Bjorklund’s Algorithm. Although Bjorklund’s Algorithm is used to create high evenness, it does not necessarily have maximal evenness. This article is also useful because it visually expresses rhythm with necklaces which are visual representations of a rhythmic string. A string is simply a list of elements, in this case pulses and steps, that make up a rhythm. The two Euclidean rhythm classes, binary and integer, are shown using *k* and *n*, which are both nonnegative integers, as different forms of height and weight of the necklace. The height is the number is the pulses and the weight is the steps or onsets.

In “Structural Properties of Euclidean Rhythms”, Bjorklund’s Algorithm is discussed more in depth showing the process step-by-step. The connection between Bjorklund’s and Euclid’s Algorithm is made and used to show different variations of Euclidean rhythms. This article is very useful because it has many mathematical proofs that help show the validity of Euclidean rhythms. The terms main and tail patterns are used to describe what Bjorklund’s algorithm is said to be composed of. The main pattern is the repeating pattern of the string whereas the tail pattern is the ending, or tail, pattern of the string.

History of Euclidean Rhythms

Euclidean Rhythms are mainly derived from the Euclidean Algorithm where the goal is to evenly distribute beats and rests (Toussaint 2005). The Euclidean algorithm comes from the famous “Euclid’s Elements” which is a collection of books used to explain different mathematical occurrences related to geometry and number theory. In book seven of his thirteen book installation, Euclid explains the concept of finding the greatest common divisor (GCD) of two numbers, or measures as he stated. The process of finding the greatest common divisor consisted of repeating subtraction. In Euclid’s illustration, which is shown in Figure 1, he expresses the line segments, AB and CD as different numbers that are not co-prime. Meaning that the GCD of the two numbers does not equal one. If CD fits into AB, with a remainder of the same size measurement, then CD is the greatest common divisor since there cannot be a divisor bigger than one of the numbers. Meaning that if the length of CD is “taken away” (or subtracted) from the measurement of AB and there is only the length of CD left over, CD is the GCD. If this is not the case, then the GCD would be found by repeatedly subtracting the smallest number from the largest number until there is no longer a remainder.



Figure 1: diagram of Euclid’s Algorithm, explaining GCD

Let AB be one measurement and let CD be a different measurement that is not co-prime with AB. In Figure 1, CD would not be considered the greatest common divisor because there is a remainder which will be called AE. If you then subtract AE (which is the length of DF) from CD, you are left with a remainder of CF. As done previously, CF has to be subtracted from AE resulting in the length of CF remaining (which is equivalent to length of AG). Therefore, CF is the greatest common divisor of AB and CD (Euclid 2002).

Now let’s express this concept numerically by attempting to find the GCD of 10 and 6. Figure 2 shows that 6 cannot be the greatest common divisor because there is a remainder of 4. Therefore, the subtraction process continues by subtracting 4 from 6. It appears that 4 is not the GCD either because there is a remainder of 2. Once the smaller number, 2, is subtracted from the larger number, 4, the remainder is 2. Thus, 2 is the greatest common divisor of 6 and 10 (Demaine et al 429-454).

A close up of a clock

Description automatically generated

Figure 2: numerical example of finding the GCD of two numbers

Bjorklund’s Algorithm is also important to the composition of Euclidean rhythms. This algorithm was first used to discuss timing systems in neutron accelerators, usually expressed using binary code (0s and 1s) and uses subtraction to evenly disperse the “1”s (*k*) and “0”s (*n*-*k*). For this algorithm, *n* is considered to be the number of time intervals and *k* is the number of steps. The first step of Bjorklund’s Algorithm is to determine how many zeroes there will be. The most popular example is having *n*=13 and *k*=5. To find the total number of zeroes, simply subtract *k* from *n* which is 8 in this example (Step 1). So, the total time is 13, there are five “1”s and eight “0”s. Now the process of distributing the “1”s and “0”s begins. First, place a “0” behind every “1” in the sequence (Step 2). After this step, there are still three “0”s left over which are then placed behind the first three sets of “10”s (Step 3). The new sequence then has two “10”s left over and in a similar fashion, the two “10”s are placed behind the first two “100”s (Step 4). This process continues until there is only one sequence left as a remainder, as is the case in the example shown in Figure 3, or there are no more ”0”s (Demaine et al 429-454). The sequence [10010] would be called the main pattern because it is repeating and [100] is the tail pattern because it is the extra sequence (Gomez-Martin and Toussaint 1-14).

1. [1 1 1 1 1 0 0 0 0 0 0 0 0]
2. [10] [10] [10] [10] [10] [0] [0] [0]
3. [100] [100] [100] [10] [10]
4. [10010] [10010] [100]

Figure 3: step-by-step process of Bjorklund’s Algorithm

Types of Euclidean Rhythms

Figure 3 shows how the Bjorklund algorithm is used to evenly distribute the 1’s and 0’s but in the case of Euclidean rhythms, it is used to represent steps and rests. Instead of *n* being the number of time intervals, it is the total number of beats in the sequence. In the example shown in Figure 3, the “1”s are replaced with “x”s and the “0”s are replaced with “.”s. The sequence now reads

[x . . x . x . . x . x . .] which is denoted E(5,13). This is a type of Euclidean rhythm associated with Macedonian music (Demaine et al 429-454). There are a number of famous world genres that can be created using Euclidean rhythms that will be discussed in this paper.

In order to form these different genres, there must be different types of rhythms. In “Euclidean Rhythms: An Investigation into the Structure of Maximal and General Evenness of Rhythms” by Tao Gaede, Euclidean rhythms are split into two different classes: one that produces maximal evenness and rhythms that are strictly derived from Bjorklund’s Algorithm. Even though Bjorklund’s Algorithm gives high evenness, it might not necessarily be maximal evenness. Simply put, high evenness does not mean that the rhythm is as even as it ultimately could be. Within the first class, there are two main categories: binary and integer. These all produce maximal evenness but the process of achieving it is different. All of the categories are explained through “necklace” examples using *k* and *n* as length and weight, respectively.

For binary necklace rhythms, *k* is length and *n* is weight which could also be viewed as the total number of beats and number of steps, respectively. The “0”s imply no event occurrence while the 1’s indicate an event occurrence. Using the example from the article where *k*=18 and *n*=13, there are two different necklaces that can be produced and they are shown in Figure 4.

1. (1,0,1,1,0,1,1,1,0,1,1,1,0,1,1,0,1,1)
2. (1,1,1,1,0,0,1,1,0,0,1,1,1,0,1,1,1,1)

Figure 4: two examples of the *k*=18 and *n*=13 binary rhythm necklace

So Figure 4.1, would be expressed as [x . x x . x x x . x x x . x x . x x] using the method above with the “x”s and “.”. This shows that there are 13 events or steps and 5 rests.

For integer necklace rhythms, *n* is the length and *k* is the weight and each element expresses the number of time units between consecutive event occurrences meaning the number of pulses between steps. Using the example from the article when *k*=18 and *n*=13, the two types of necklaces that can be produced are shown in Figure 5.

1. (2,1,2,1,1,2,1,1,2,1,2,1,1)
2. (1,1,1,3,1,3,1,1,2,1,1,1,1)

Figure 5: two examples of the *k*=18 and *n*=13 integer rhythm necklace

Figure 5.1, would be expressed as [x . x x . x x x . x x x . x x . x x]. Starting at the first onset, the 2 would represent how many pulses it takes until the next onset or event (Gaede 2018).

Brad Osborn, discovered four distinct types of Euclidean rhythms through analyzing the band Radiohead. The first type is when *k* (the number of steps) is a factor of *n* (the total number of beats) which means *k* divides *n* evenly. An example of this type would be E(4,8) which would be visually expressed as [x . x . x . x .]. The second type is when *k* is not a factor of *n* and *k* and *n* are not co-prime, meaning there is some integer smaller than *k* and *n* that divides both numbers evenly (like the example given with the Euclidean algorithm). An example of this type would be E(4,10) which is expressed as [x . . x . . x . x .]. The third type is when *k* is not a factor of *n*, *k* and *n* are co-prime and *k* is not a factor of [n±1]. An example of this type is E(7,16) which is represented as [x . . x . . x . x . x . x . x .]. The final type discussed is similar to the third but *k* is a factor of [n±1]. The example given for this rhythm is E(5,16) which can be shown as [x . . . x . . x . . x . . x . .] (Osborn 81-105). As shown in the paper, there are many different types of Euclidean rhythms. There were over 40 genres of music found through producing different Euclidean rhythms. Figure 6 shows a few popular genres that can be found.

E(5,12)=[x . . x . x . . x . x .] a South African rhythm

E(5,11)=[x . x . x . x . x . .] a classical rhythm

E(7,16)=[x . . x . x . x . . x . x . x .] a Brazilian rhythm

Figure 6: three popular Euclidean rhythm necklaces from around the world

Euclidean Strings and Euclidean Rhythms

Euclidean strings, like Euclidean rhythms, are derived from Euclid’s Algorithm. They were first discovered through the study of combinatorics- a branch of mathematics that studies the combinations of elements. Strings, denoted as P, are considered Euclidean if increasing P0 (the initial element of the string) by one and decreasing Pn-1 (the last element of the string) by one yields a new string, denoted as τ(P). It must also be a rotation of P so it’s a part of the same necklace. A rotation is the shift of the elements in the string to the left. Euclidean strings cannot be done in binary form because the binary form expresses the string in “1”s and “0”s. When Euclidean strings are shown, the strings must be represented by their “adjacent-inter-onset-duration-interval-vectors” (or interval vectors form for short). It can also be seen as the integer necklace rhythm form from earlier in the paper.

For example, E(4,9) = [x . x . x . x . .] would be represented as (2223) showing the number of pulses until the next onset. τ(2223) = 3222 = p3(P). The 3 represents the number of rotations applied to the string. Since τ(2223) is a new string that is a rotation of P, it means that P is a Euclidean string and Euclidean rhythm.

The reversal string (or mirror image) of P, denoted R(P), is a special type of Euclidean string. The reversal Euclidean string occurs when P is written backwards.

Euclidean Rhythm and String Example:

E(3,7) = [x . x . x . .] = (223) Bulgarian Folk

τ(3,7) = [x . . x . x .] = (322)

Euclidean Rhythm and Reverse String Example:

E(5,12) = [x . . x . x . . x . x .] = (32322) South Africa

R(5,12) = [x . x . x . . x . x . .] = (22323)

Euclidean Rhythm that is neither a Euclidean String nor Reversal Euclidean String:

E(5,8) = [x . x x . x x .] = (21212) West Africa

τ(5,8) = [x . x . x x . x x] = (22121) doesn’t count as a Euclidean string because it’s a different rhythm necklace. The original Euclidean rhythm has only five onsets while τ(5,8) has six so it is a different rhythm.

R(5,8) = [x . x x . x x .] = (21212) does not count as a Euclidean string because it is the same string as the original Euclidean rhythm (Toussaint 2005).

Euclidean Rhythm Operations

In the article “Interlocking and Euclidean Rhythms”, Francisco Gómez-Martína, Perouz Taslakian and Godfried Toussaint discuss operations that can be applied to Euclidean rhythms to change their properties. They are used to compose rhythmic cannons and improvise solos. There are three main operations described: complementation, alteration and decomposition. Complementation is when a rhythm’s elements are switched. For example, the original Cuban cinquillo rhythm is read as [x . x x . x x .]. The complementary rhythm is [. x . . x . . x] which is also a rotation of the Cuban tresillo rhythm [x . . x . . x .]. For complementation, all the steps are switched to rests and the rests are switched to steps.

Starting from the *j*th onset, the alternation operation keeps every *c*th onset of the Euclidean rhythm and replaces the remaining onsets with rests. The alternation operation is expressed as Aj,c(R) where R stands for rhythm. This operation is best explained through an example.

E(7,17) = [x . . x . x . . x . x . . x . x .]

A0.2(7,17) = [x . . . . x . . . . x . . . . x .]

The alternate version of this Euclidean rhythm was found by starting at the onset in position 0 (the initial one) and keeping every 2nd onset after that. Turning the second, fourth and sixth onset into rests while keeping the first, third, fifth and seventh onset the same. An alternation has the potential to eliminate a rhythm’s Euclidean characteristics. The final operation, decomposition, involves merging two rhythms. Let R1 and R2 denote two rhythms with the same number of pulses and let R be the union of those two rhythms (Gómez-Martín 2009).

Example:

R1= [x . . . x x . x . . . x]

R2= [x . x . x . . x . x . .]

R= [x . x . x x . x . x . x]

Conclusion

In this paper, Euclidean rhythms’ applications and history is discussed. The different types were explained and the operations that can be applied to them. Euclidean rhythms are a prime example of the longevity math and music has. No matter how “old” a topic is, it can still be related to a current subject. Since Euclidean rhythms are a newer topic, I found it difficult to find as much information on it as I wanted. Also, trying to understand the information was hard at times since some of it was more complex than I anticipated. With that being said, one day I do hope to come back to this topic once I have a better understanding of some of the mathematical concepts so that I can add my own input to the discussion. During my research, I noticed the lack of evidence of Euclidean rhythms expressing any modern genres like Pop or Hip Hop. I would like to further investigate my topic to see if it is possible to create these types of rhythms are possible to generate with Euclidean rhythms.

Work Cited

E. D. Demaine, F. Gomez-Martin, H. Meijer, D. Rappaport, P. Taslakian, G. T. Toussaint, T. Winograd and D. R. Wood, "The distance geometry of music," Computational Geometry: Theory and Applications, no. 42, pp. 429-454, 2009.

Euclid, , Thomas L. Heath, and Dana Densmore. *Euclid's Elements: All Thirteen Books Complete in One Volume : the Thomas L. Heath Translation*. Santa Fe, N.M: Green Lion Press, 2002. Print.

G. Toussaint, "The Euclidean algorithm generates traditional musical rhythms," 2005.

Gómez-Martín, Francisco, et al. “Interlocking and Euclidean Rhythms.” *Journal of Mathematics and Music*, vol. 3, no. 1, 2009, pp. 15–30., doi:10.1080/17459730902916545.

Gomez-Martin, P. Taslakian and G. Toussaint, "Structural properties of Euclidean rhythms," Journal of Mathematics and Music, vol. 3, no. 1, pp. 1-14, 2009.

Osborn, Brad. 2014. “Kid Algebra: Radiohead’s Euclidean and Maximally Even Rhythms.” *Perspectives of New Music* 52 (1): 81–105.

T. Gaede, "Euclidean rhythms: an investigation into the structure of maximal and general evenness of rhythms," 2018.